[8] S. Adachi, "Study on the guiding mechanism of whistler radio waves," *Radio Sci.*, vol. 69 D, pp. 493–502, Apr. 1965.
[9] T. C. Lim and G. W. Farnell, "Character of psuedo surface waves on anisotropic crystals," *J. Acoust. Soc. Amer.*, vol. 45, pp. 845–851, Apr. 1969.
[10] A. Takayanagi, K. Yamanouchi, and K. Shibayama, "Piezo-electric leaky surface wave in $LiNbO_3$," *Appl. Phys. Lett.*, vol. 17, pp. 225–227, Sept. 1970.
[11] D. Marcuse, "TE modes of graded-index slab waveguides," *IEEE J. Quantum Electron.*, vol. QE-9, pp. 1000–1006, Oct. 1973.

# Analysis of Intermodulation Noise in Frequency Converters by Volterra Series

RICHARD B. SWERDLOW, MEMBER, IEEE

***Abstract*—Frequency converters produce intermodulation noise in the desired signal band which may be a serious problem for communications systems using amplitude modulation. In this paper, we introduce the Volterra series with time-varying kernels to treat intermodulation in frequency converters with one two-terminal nonlinearity. The method gives exact results up to the order calculated (third order here) for any nonlinearity expressable as a power series, will treat frequency dependence in the nonlinearity as well as the terminations, and leads to a convenient algorithm for computer calculation. The mathematics provides a physical picture of intermodulation of a specific order as being produced by the modulation of lower order products through the nonlinearity. In fact, the solution for a given order of intermodulation currents or charges is the solution of a set of linear equations where the driving functions are intermodulation currents of lower order.**

**A program has been written for the specific but important case of an abrupt junction varactor upconverter. Results for an upconverter that might be used for single-sideband operation in the common carrier microwave band show that the largest contribution to intermodulation comes from currents which are at the sum and difference frequencies of the input (IF) signal, corresponding to currents above the input port in frequency and currents in the bias circuitry.**

**This paper documents previously unpublished work (1972) done as part of the exploratory study of single-sideband modulation on long-haul microwave radio transmission.**

## I. Introduction

At least four distinct procedures have been used to calculate the intermodulation distortion of upconverters. Anderson and Leon [1] computed bounds on the intermodulation distortion by an approximate solution to the integral equation describing the converter. Perlow and Perlman [2] related gain compression of a single tone to the magnitudes of distortion products from two tones. Schwarz and Nelson [3] assumed current flow at input, output, pump, and bias circuitry and solved for the intermodulation by substituting into a power series for the diode nonlinearity. Gardiner and Ghobrial [4] express the charge as a series in auxilliary functions which can be found from a set of first-order differential equations. Rice [5] has related this procedure to solution by the Volterra series.

In the present method, the solution is found as a time-varying Volterra series [6]. The first term of the series leads to the linear theory of frequency converters similar to that of Penfield and Rafuse [7]. Thus the formulae and computer algorithms found here are applicable to all types of frequency converters: upconverters to upper and lower sidebands, upconverters with idlers, the equivalent downconverters and parametric amplifiers. The basic limitation is to two-terminal nonlinearities. Formulae developed here are valid for a nonlinear resistance and capacitance in series or the dual situation of parallel nonlinear conductance and inductance. Any linear time-invariant termination is permitted at any frequency where currents may flow; indeed, an important conclusion is that for an upconverter suitable for heterodyne repeaters in the 6-GHz common-carrier band, currents at the sum and difference frequencies of the IF are the substantial contributors to third-order intermodulation distortion at the output. No attempt is made to find analytic solutions to the intermodulation. A strength of the analysis is that it results in an algorithm well-suited to computer calculation; namely, sets of linear equations. The Volterra kernels found permit calculation of gain compression, intermodulation due to multitone inputs, and noise inputs.

## II. Circuit to be Studied

Fig. 1 is a schematic representation of an upconverter using a diode as the nonlinearity. The nonlinearity is two terminal and this analysis is restricted to this case; more than one independent current flowing through nonlinearities would require a multidimensional Volterra series. Thevinen's and Norton's theorems were used to lump the linear portion of the circuit, which includes the bias, pump, output, etc., into the two-terminal impedance as

Manuscript received May 9, 1977; revised October 14, 1977.
The author is with Bell Laboratories, North Andover, MA 01845.

0018-9480/78/0400-0305$00.75 

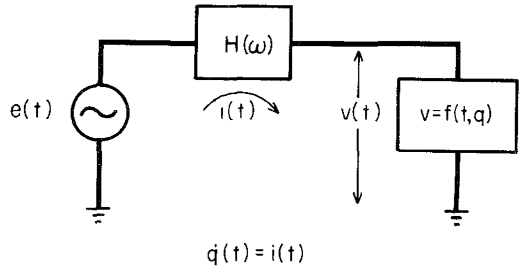


Fig. 1. Equivalent circuit for converter.

seen by the diode. As long as all the pump current flows in the diode, the pump source and diode may be replaced by a time-varying nonlinear element as shown. The pump need not be large compared to the signal current to make this substitution, but if the current at the pump frequency produced by the signal becomes a substantial portion of the pump current, then the accuracy of this replacement suffers accordingly. The formulae in Section IV permit calculation of the pump current due to the signal. In Appendix I, the power series for the varactor as a time-varying element is developed as

$$v(t)=\sum_{l=0}^{\infty} a_l(t)q^l(t)+\sum_{l=0}^{\infty} b_l(t)\dot{q}^l(t) \tag{2-1}$$

where

$$a_l(t)=\sum_{k=0}^{\infty} a_{lk}\sin k\omega_p t \tag{2-2}$$

$$b_l(t)=\sum_{k=0}^{\infty} b_{lk}\cos k\omega_p t \tag{2-3}$$

where $\omega_p$ is the pump radian frequency, $a_{lk}$, $b_{lk}$ are Fourier coefficients for the nonlinear resistance and capacitance, respectively.

## III. Solution Form for $q(t)$

$q(t)$ will be solved in terms of $e(t)$ using the Volterra series [6], which permits specifications of the terminations in terms of their frequency response $H(\omega)$. Different forms of $H(\omega)$ may be chosen for each termination, in analytic or tabular form. The expansion for $q(t)$ is

$$\begin{aligned} q(t)=&\int_{-\infty}^{t} g_1(t;\tau_1)e(\tau_1)\,d\tau_1\\ &+\int_{-\infty}^{t}\int_{-\infty}^{t} g_2(t;\tau_1,\tau_2)e(\tau_1)e(\tau_2)\,d\tau_1\,d\tau_2\\ &+\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} g_3(t;\tau_1,\tau_2,\tau_3)e(\tau_1)e(\tau_2)\\ &\cdot e(\tau_3)\,d\tau_1\,d\tau_2\,d\tau_3+\cdots \end{aligned} \tag{3-1}$$

$$g_n(t;\tau_1,\tau_2,\cdots\tau_n)=0, \qquad \text{for } \tau_i>t, \text{ all } i \tag{3-2}$$

where $g_1(t;\tau)$ is the time-varying system function of Zadeh [8], and $g_2$, $g_3$ are extensions of his formulation for the higher order terms. The solution is obtained in terms of the transforms of these kernels. For the $n$th-order kernel

$$\begin{aligned}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} & g_n(t;\tau_1,\tau_2,\cdots\tau_n)\\ &\cdot e^{j(\omega_1\tau_1+\omega_2\tau_2+\cdots+\omega_n\tau_n)}d\tau_1\cdots d\tau_n\\ &=G(t;\omega_1,\omega_2,\cdots\omega_n)e^{j(\omega_1+\omega_2+\cdots+\omega_n)t}\end{aligned} \tag{3-3}$$

which reduces to Zadeh's definition of the time-varying system function $G(t;\omega)$, for $n=1$. Since we are dealing with a periodically time-varying network, we may expand $G$ in a Fourier series

$$G(t;\omega_1,\omega_2,\cdots\omega_m)=\sum_{n=-\infty}^{\infty} G_n(\omega_1,\omega_2,\cdots\omega_m)e^{jn\omega_p t}. \tag{3-4}$$

The $G_n$ are the Fourier coefficients of the time-varying system function $G(t;\omega_1,\cdots\omega_m)$, which is itself the transform of $g_m(t;\omega_1,\cdots\omega_m)$. Since all the kernels $g_m(t;\omega_1,\omega_2\cdots\omega_m)$ are real, we can readily establish the convenient relationship

$$G_n(\omega_1,\omega_2,\cdots\omega_m)=G^*_{-n}(-\omega_1,-\omega_2,\cdots,-\omega_m) \tag{3-5}$$

where the asterisk refers to complex conjugate. Although not explicitly shown, the $G_n$ are also functions of $n\omega_p$.

The solution will be obtained in terms of these $G_n(\omega_1,\omega_2,\cdots\omega_m)$ and they have a convenient operational meaning for the upconverter. For example, for a transformed voltage excitation $E(\omega)$, the transform of the third-order intermodulation charge is, excluding harmonics of the pump frequency,

$$\begin{aligned}Q(\omega_1,\omega_2,\omega_3)=\sum_{n=-1}^{1} & G_n(\omega_1,\omega_2,\omega_3)\\ &\cdot e^{jn\omega_p t}E(\omega_1)E(\omega_2)E(\omega_3)\end{aligned} \tag{3-6}$$

so that $G_1(\omega_1,\omega_2,\omega_3)$ is a transconductance relating a voltage excitation at the input port to a third-order intermodulation charge at the upper sideband. $G_0(\omega_1,\omega_2,\omega_3)$ gives the reflected intermodulation charge at the input and $G^*_{-1}(\omega_1,\omega_2,\omega_3)$ refers to the lower sideband. The first-order terms develop the linear behavior of the converter. For example, the power gain is

$$g=\frac{p_{\text{out}}}{p_{\text{in}}}=\frac{E^2|j(\omega_p+\omega)G_1(\omega)|^2\,\text{Re}\,[H(\omega_p+\omega)]/2}{E^2\,\text{Re}\,[j\omega G_0(\omega)]/2} \tag{3-7}$$

where $g$ is the power gain, $p_{\text{out}}$, $p_{\text{in}}$, the power dissipated in the output and input, respectively, and $H(\omega)$ the termination impedance at $\omega$. The $j\omega$ terms convert the charges to currents. The linear theory is close to that of Penfield and Rafuse. For example, see Section 5.2 of [7].

## IV. General Solution for the Two-Terminal Frequency Changer

Kirchoff's voltage law for the circuit of Fig. 1 reads

$$e(t)=\int_{-\infty}^{\infty} h(t-\alpha)\dot{q}(\alpha)\,d\alpha+\sum_{l=1}^{\infty} a_l(t)\dot{q}^l(t)+\sum_{l=1}^{\infty} b_l(t)q^l(t) \tag{4-1}$$

where the coefficients $a_l$ and $b_l$ were introduced earlier and are related to diode characteristics in Appendix I, and the dot over $q(t)$ refers to time differentiation. We can solve this equation using Flake's procedure, that is, insert the expression for $q(t)$ in (3-1) into (4-1), expand and equate functionals while retaining functionals to the third order.

We need the time derivative of $q(t)$ expressed as the functional series in (3-1) to do this. Because of the causality condition (3-2) on the kernels of the series, additional terms will appear for the boundary conditions. For example, the derivative of the second-order kernel will be

$$\frac{d}{dt}\int_{-\infty}^{t}\int_{-\infty}^{t} g_2(t;\tau_1,\tau_2)e(\tau_1)e(\tau_2)\,d\tau_1\,d\tau_2 = \int_{-\infty}^{t}\int_{-\infty}^{t} \frac{\partial}{\partial t} g_2(t;\tau_1,\tau_2)e(\tau_1)e(\tau_2)\,d\tau_1\,d\tau_2 + e(t)\int_{-\infty}^{t} g_2(t;\tau_1,t)e(\tau_1)\,d\tau_1 + e(t)\int_{-\infty}^{t} g_2(t;t,\tau_2)e(\tau_2)\,d\tau_2. \tag{4-2}$$

We can avoid the explicit appearance of these boundary terms by allowing the derivative of the kernal to contain impulses. Thus the expansion of (4-2) would contain only the first term; but $\partial g_2(t;\tau_1,\tau_2)/\partial t$ might contain terms such as $g_2(t;t,\tau_2)\delta(t-\tau_1)$. This permits us to extend the limits of integration over all time. The causality of the kernels will be assured by the nature of the solution. As we will see, the first-order kernels will be determined by equations linear in $h(t)$, the impulse response of the linear circuit elements. Thus the first-order kernels must be causal since $h(t)$ is. The second-order kernels are found from equations linear in $h$, and driven by combinations of first-order kernels. Thus the second-order kernels are causal. In the same way, all higher order kernels will be causal. Therefore, all integrals will be over all time unless otherwise noted. The equations found by Flake's procedure are

for functionals $\int e(\tau_1)\,d\tau_1$:

$$\int d\alpha h(t-\alpha)\frac{\partial}{\partial \alpha} g_1(\alpha;\tau_1) + a_1(t)\frac{\partial}{\partial t} g_1(t;\tau_1) + b_1(t)g_1(t;\tau_1) = \delta(t-\tau_1) \tag{4-3a}$$

for $\int\int e(\tau_1)e(\tau_2)\,d\tau_1\,d\tau_2$:

$$\int d\alpha h(t-\alpha)\frac{\partial}{\partial \alpha} g_2(\alpha;\tau_1,\tau_2) + a_1(t)\frac{\partial}{\partial t} g_2(t;\tau_1,\tau_2) + b_1(t)g_2(t;\tau_1,\tau_2) + a_2(t)\frac{\partial}{\partial t} g_1(t;\tau_1)\frac{\partial}{\partial t} g_1(t;\tau_2) + b_2(t)g_1(t;\tau_1)g_1(t;\tau_2) = 0 \tag{4-3b}$$

and for $\int\int\int e(\tau_1)e(\tau_2)e(\tau_3)\,d\tau_1\,d\tau_2\,d\tau_3$:

$$\int d\alpha h(t-\alpha)\frac{\partial}{\partial \alpha} g_3(\alpha;\tau_1,\tau_2,\tau_3) + a_1(t)\frac{\partial}{\partial t} g_3(t;\tau_1,\tau_2,\tau_3) + b_1(t)g_3(t;\tau_1,\tau_2,\tau_3) + 2a_2(t)\frac{\partial}{\partial t} g_1(t;\tau_1)\frac{\partial}{\partial t} g_2(t;\tau_2,\tau_3) + a_3(t)\frac{\partial}{\partial t} g_3(t;\tau_1,\tau_2,\tau_3) + 2b_2(t)g_1(t;\tau_1)g_2(t;\tau_2,\tau_3) + b_3(t)g_3(t;\tau_1,\tau_2,\tau_3) = 0. \tag{4-3c}$$

To obtain the transforms we multiply (4-3a) by $e^{j\omega_1\tau_1}$ and integrate over $\tau_1$, use the definition of the transform of the time-varying kernel in (3-3), and its expansion in (3-4). The transforms are obtained similarly for (4-3b) and (4-3c), using

$$e^{j(\omega_1\tau_1+\omega_2\tau_2)} \quad \text{and} \quad e^{j(\omega_1\tau_1+\omega_2\tau_2+\omega_3\tau_3)}$$

respectively. The final equations are (all sums $-\infty$ to $\infty$):

$$\sum_n G_n(\omega_1)e^{jn\omega_p t}j(n\omega_p+\omega_1)H(\omega_1+n\omega_p) + \sum_i\sum_n G_n(\omega_i)\alpha(i,n)e^{j(i+n)\omega_p t} = 1 \tag{4-4a}$$

$$\sum_n G_n(\omega_1,\omega_2)e^{jn\omega_p t}\cdot j\left[n\omega_p+(\omega_1+\omega_2)\right]H\left[(\omega_1+\omega_2)+n\omega_p\right] + \sum_i\sum_n G_n(\omega_1,\omega_2)\alpha(i,n)e^{j(i+n)\omega_p t} = -\sum_m\sum_l\sum_k G_m(\omega_1)G_l(\omega_2)\cdot\beta(k,m,l)e^{j(k+l+m)\omega_p t} \tag{4-4b}$$

$$\sum_n G_n(\omega_1,\omega_2,\omega_3)e^{jn\omega_p t}\cdot j\left[n\omega_p+(\omega_1+\omega_2+\omega_3)\right]H\left[(\omega_1+\omega_2+\omega_3)+n\omega_p\right] + \sum_i\sum_n G_m(\omega_1,\omega_2)\alpha(i,n)e^{j(i+n)\omega_p t} = -2\sum_m\sum_l\sum_k G_m(\omega_1,\omega_2)G_l(\omega_3)\cdot\beta(k,m,l)e^{j(m+l+k)\omega_p t} - \sum_m\sum_l\sum_p\sum_k G_m(\omega_1)G_l(\omega_2)G_p(\omega_3)\cdot\gamma(k,m,l,p)e^{j(m+l+p+k)\omega_p t} \tag{4-4c}$$

where

$$j(\omega_1+n\omega_p)a_1(t)+b_1(t) = j(\omega_1+n\omega_p)a_{10}+b_{10} + \sum_{i=-\infty}^{\infty}\left[(\omega_1+n\omega_p)\frac{a_{1i}}{2}\operatorname{sgn}(i)+\frac{b_{1i}}{2}\right]e^{ji\omega_p t} = \sum_{i=-\infty}^{\infty}\alpha(i,n)e^{ji\omega_p t} \tag{4-5a}$$

$$\operatorname{sgn}(x) = +1, \quad x \geqslant 0$$
$$= -1, \quad x < 0$$

$$-(\omega_1 + m\omega_p)(\omega_2 + l\omega_p)a_2(t) + b_2(t)$$
$$= -(\omega_1 + m\omega_p)(\omega_2 + l\omega_p)a_{20} + b_{20}$$
$$+ \sum_{i=-\infty}^{\infty} \left[ -(\omega_1 + m\omega_p)(\omega_2 + l\omega_p)\frac{a_{2i}}{2j} \cdot \operatorname{sgn}(i) + \frac{b_{2i}}{2} \right] e^{ji\omega_p t}$$
$$= \sum_{i=-\infty}^{\infty} \beta(m,l,i)e^{ji\omega_p t}. \tag{4-5b}$$

$$\begin{bmatrix} j(\omega_1+\omega_p)H(\omega_1+\omega_p)+b_{10} & \frac{b_{11}}{2} & 0 \\ \frac{b_{11}}{2} & j\omega_1 H(\omega_1)+b_{10} & \frac{b_{11}}{2} \\ 0 & \frac{b_{11}}{2} & j(\omega_1-\omega_p)H(\omega_1-\omega_p)+b_{10} \end{bmatrix} \begin{bmatrix} G_1(\omega_1) \\ G_0(\omega_1) \\ G_{-1}(\omega_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{5-1b}$$

(In (4-4c) $\omega_1$ here is replaced by $\omega_1+\omega_2$, $\omega_2$ by $\omega_3$.)

$$-j(\omega_1 + m\omega_p)(\omega_2 + l\omega_p)(\omega_3 + p\omega_p)a_3(t) + b_3(t)$$
$$= -j(\omega_1 + m\omega_p)(\omega_2 + l\omega_p)(\omega_3 + p\omega_p)a_{30} + b_{30}$$
$$+ \sum_{i=-\infty}^{\infty} \left[ -(\omega_1 + m\omega_p)(\omega_2 + l\omega_p)(\omega_3 + p\omega_p) \cdot \frac{a_{3i}}{2} + \frac{b_{3i}}{2} \right] e^{ji\omega_p t}$$
$$= \sum_{i=-\infty}^{\infty} \gamma(m,l,p,i)e^{ji\omega_p t}. \tag{4-5c}$$

The difference in coefficients for $a(t)$ and $b(t)$ is that $a$ is taken as a sine series and $b$ as a cosine series because they will be in quadrature during pumping. The $b$ coefficients are related to the static properties of the varactor diodes in Appendix I.

The only restrictions on this solution are that the nonlinearity be two terminal and the general requirements for a Volterra solution be satisfied. Thus up or down conversion and any linear system function and diode characteristic are allowed.

## V. Solution to Abrupt Junction Four-Frequency Upconverter

Because of its practical importance and analytic simplicity, we will continue the analysis with an upconverter using a single abrupt junction varactor. All frequencies below the second harmonic of the pump for which conduction is possible will be considered.

For the abrupt junction case, the results of Appendix I show all $b_{ij}$ beyond $b_{20}$ to be zero. Thus

$$\gamma(i,m,l,p) = 0, \quad \text{all indices.} \tag{5-1a}$$

The only nonzero coefficients of (4-5) are

$\alpha(0,n) = b_{10}$, the elastance under pumped conditions

$\alpha(1,n) = \frac{b_{11}}{2}$, the conversion elastance

$\beta(0,m,l) = -b_{20}$, coefficient of nonlinearity.

### A. First-Order Equation

We will write out (4-4a). Only conduction below the second harmonic of the pump will be considered; so all $G_n$, for which $n \geqslant 2$, are zero. Thus we need write equations only for $e^{j\omega_p t}$, $e^{j0t} = 1$, and $e^{-j\omega_p t}$. Expressing these in matrix form,

$$Z(\omega_1)G(\omega_1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The zeros in the matrix normally are filled by coefficients of second-order capacitance variations for other than an abrupt junction varactor; this situation is discussed by Korpel and Ramaswamy [9]. $b_{11}$, the conversion elastance, is $2S_1$ in the nomenclature of [7]. If the lower sideband does not conduct, then $H(\omega_1-\omega_p) \rightarrow \infty$, $G_{-1}(\omega_1) = 0 = G_1^*(-\omega_1)$. The selection of $\omega_1$ is arbitrary; if it is selected to be above the pump, $G_{-1}(\omega_1)$ describes downconversion and subsequent results for second- and third-order distortion are for downconversion.

### B. Second-Order Terms

Equation (4-4b) will yield three equations for the harmonics $e^{j\omega_p t}$, 1, and $e^{-j\omega_p t}$ corresponding to $n = 1, 0, -1$, just as for the first-order case. The matrix equation is now

$$Z(\omega_1+\omega_2)G(\omega_1,\omega_2) = Y_2 \tag{5-2}$$

where

$$Y_2 = -b_{20} \cdot \begin{bmatrix} G_0(\omega_1)G_1(\omega_2) + G_1(\omega_1)G_0(\omega_2) \\ G_{-1}(\omega_1)G_1(\omega_2) + G_0(\omega_1)G_0(\omega_2) + G_1(\omega_1)G_{-1}(\omega_2) \\ G_{-1}(\omega_1)G_0(\omega_2) + G_0(\omega_1)G_{-1}(\omega_2) \end{bmatrix}. \tag{5-3}$$

These equations demonstrate clearly how intermodulation of a given order may be viewed as being produced by intermodulation of charges of lower order through the diode nonlinearity, as suggested by Gardiner and Ghobrial. Thus if two tones are imagined at the input port at frequencies $\omega_1$ and $\omega_2$, then (5-3) shows that intermodulation at the pump ($n = 1$) is produced by fundamentals from the input $G_0(\omega_1)$ and the output $G_1(\omega_2)$. The frequency of the product will be at $\omega_p+\omega_1+\omega_2$. Depending on the signs of $\omega_1$ and $\omega_2$, $G_1(\omega_1,\omega_2)$ may represent second-order tones about the pump $\omega_p \pm (\omega_1-\omega_2)$, above the upper sideband $\omega_p+(\omega_1+\omega_2)$, or below the lower sideband $\omega_p-(\omega_1+\omega_2)$. A similar description holds for $G_0(\omega_1,\omega_2)$. In this case, intermodulation current will flow in the bias circuitry at $\omega_1-\omega_2$. In the limit as $|\omega_1|$ and $|\omega_2|$ approach each other, the terms in (5-3) for $n=0$ represent static charge on the diode due to rectification of the signal charges.

*C. Third-Order Terms*

Equation (4-4c) yields

$$\boldsymbol{Z}(\omega_1+\omega_2+\omega_3)\boldsymbol{G}(\omega_1,\omega_2,\omega_3)=\boldsymbol{Y}_3 \tag{5-4}$$

where

$$\boldsymbol{Y}_3=-2b_{20}\begin{bmatrix} G_0(\omega_1,\omega_2)G_1(\omega_3)+G_1(\omega_1,\omega_2)G_0(\omega_3) \\ G_{-1}(\omega_1,\omega_2)G_1(\omega_3)+G_0(\omega_1,\omega_2)G_0(\omega_3) \\ +G_1(\omega_1,\omega_2)G_{-1}(\omega_3) \\ G_{-1}(\omega_1,\omega_2)G_0(\omega_3)+G_0(\omega_1,\omega_2)G_{-1}(\omega_3)\end{bmatrix}. \tag{5-5}$$

This shows that only second-order charges produce third-order intermodulation in an abrupt junction converter; no terms such as $G_1(\omega_1)G_{-1}(\omega_2)G_0(\omega_3)$ are present. This explains the importance of terminations for second-order charges in reducing third-order intermodulation in frequency converters of this type.

## VI. Three-Tone Input

A common test of intermodulation is to excite the device under test with three in-band tones at frequencies $\omega_A$, $\omega_B$, and $\omega_C$, and measure the third-order in-band tone at $\omega_A-\omega_B+\omega_C$, the largest of in-band third-order tones. For a converter, this tone is at $\omega_p+\omega_A-\omega_B+\omega_C$. A useful measure of third-order distortion, called the third-order intermodulation coefficient [10], is defined as

$$M_{3eq}=P_{A-B+C}-(P_A+P_B+P_C) \tag{6-1}$$

where $P_X$ is the power in dBm of the tone at frequency $X$. This measure is a constant with varying power levels, as long as the device is third order, so it can be used as a property of the device. The subscript "*eq*" refers to the use of the in-band tone $A-B+C$, rather than the usual definition for $A+B+C$ in [10].

To compute this quantity, we need the power for the third-order charges produced by the voltage excitation

$$e(t)=E\cos(\omega_A t+\theta_A)+E\cos(\omega_B t+\theta_B)+E\cos(\omega_C t+\theta_C) \tag{6-2}$$

where $E$ is the peak voltage amplitude, $\omega_A$, $\omega_B$, $\omega_C$ are the radian frequencies of the tones, and $\theta_A$, $\theta_B$, $\theta_C$ are the arbitrary phases. The excitation in the frequency domain will be $E(\omega_1)E(\omega_2)E(\omega_3)$, where, for example,

$$E(\omega_1)=\frac{2\pi E}{2}\left[\delta(\omega_1-\omega_A)+\delta(\omega_1+\omega_A)+\delta(\omega_1-\omega_B)+\delta(\omega_1+\omega_B)+\delta(\omega_1-\omega_C)+\delta(\omega_1+\omega_C)\right] \tag{6-3}$$

where the phases have been dropped for clarity. The third-order current found from the in-band terms of $G_1(\omega_1,\omega_2,\omega_3)E(\omega_1)E(\omega_2)E(\omega_3)$ is

$$i_3(t)=\frac{E^3}{4}|j(\omega_p+\omega_A-\omega_B+\omega_C)G_1(\omega_A,-\omega_B,\omega_C)| \cdot \cos\left[(\omega_A-\omega_B+\omega_C)t+\phi_{A,-B,C}\right]$$
$$+\text{like forms in } \omega_A,\omega_C,-\omega_B;\ -\omega_B,\omega_A,\omega_C;\ -\omega_B,\omega_C,\omega_A;\ \omega_C,\omega_A,-\omega_B;\ \omega_C,-\omega_B,\omega_A \tag{6-4}$$

where $\phi_{A,-B,C}$ is the angle of $G_1(\omega_A,-\omega_B,\omega_C)$, and

$$M_{3eq}=10\log_{10}\frac{|j(\omega_p+\omega_A-\omega_B+\omega_C)\bar{G}_1(\omega_A,\omega_B,\omega_C)|^2\,\mathrm{Re}\left[H(\omega_A-\omega_B+\omega_C)\right]}{2^2\prod_{i=A,B,C}|j(\omega_p+\omega_i)G_1(\omega_i)|^2\,\mathrm{Re}\left[H(\omega_i)\right]} \tag{6-5}$$

where

$$\bar{G}_1(\omega_A,\omega_B,\omega_C)=\sum_{\omega_A,-\omega_B,\omega_C} G_1(\omega_A,-\omega_B,\omega_C) \tag{6-6}$$

and the sum is over the six permutations of frequencies in (6-4).

## VII. Computational Results for the Abrupt Junction Varactor Upconverter

A computer program has been written for the abrupt junction varactor which computes the Volterra kernels up to order three according to the equations in Section V. The program solves (5-1), (5-2), and (5-4) in that order, using results of (5-1) in (5-2) and so on.

The particular design parameters considered were for an upconverter that might be used in a microwave radio system used in the 6-GHz common carrier band for telephony. The parameters are given in Table I. Questions of interest were what terminations were particularly important for linearity for this converter, and how small $M_{3eq}$ might be made. Typical values of $M_{3eq}$ for currently used converters range from $-20$ dB to $-30$ dB.

Results are plotted in Fig. 2 and 3. In Fig. 2, each curve is the $M_{3eq}$ obtained by changing the termination noted on the curve to the value on the abscissa, while all other terminations labelled "varied" on Table I were open circuited. For example, the curve labelled "bias circuit" in Fig. 2 gives the $M_{3eq}$ obtained for a converter whose bias circuit resistance was varied from 50 Ω to 50 kΩ while the resistances seen by the lower sideband, the pump circuit, the sum frequencies of the IF input, the sum frequencies

TABLE I
PARAMETERS FOR 6-GHz UPCONVERTER

| QUANTITY | VALUE | UNITS |
|---|---|---|
| Diode capacitance | 0.5 | picofarads |
| Pump current | 200 | milliampers |
| Bias voltage | -20 | volts |
| Input termination* | 05 | kilohms |
| Upper sideband (output)* | .05 | kilohms |
| Lower sideband | varied | kilohms |
| Pump circuit* | varied | kilohms |
| Bias circuit | varied | kilohms |
| Second orders above input | varied | kilohms |
| Second orders above upper sideband | varied | kilohms |
| Second orders below lower sideband | varied | kilohms |
| Pump frequency | 6 | gigahertz |
| Highest frequency on input | .089 | gigahertz |
| Lowest frequency on input | 059 | gigahertz |
| Input frequency 1 | .088 | gigahertz |
| Input frequency 2 | .074 | gigahertz |
| Input frequency 3 | .060 | gigahertz |

*The program automatically uses a conjugate match to the diode for these ports.

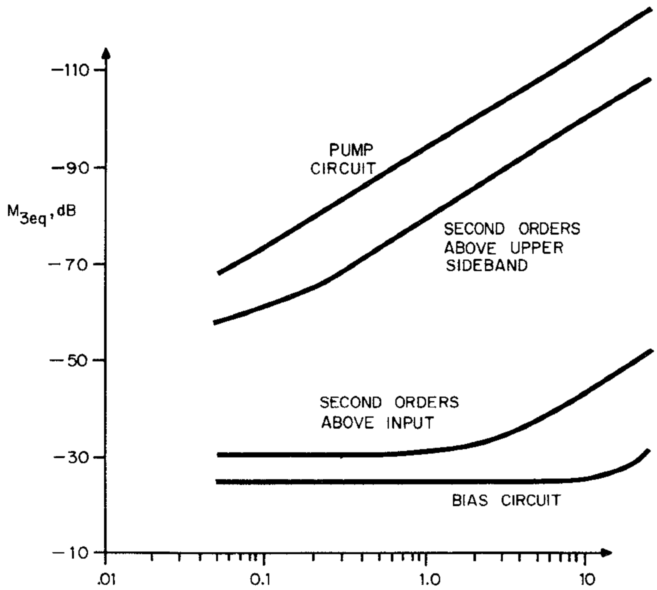


Fig. 2. $M_3$eq variations with terminations.

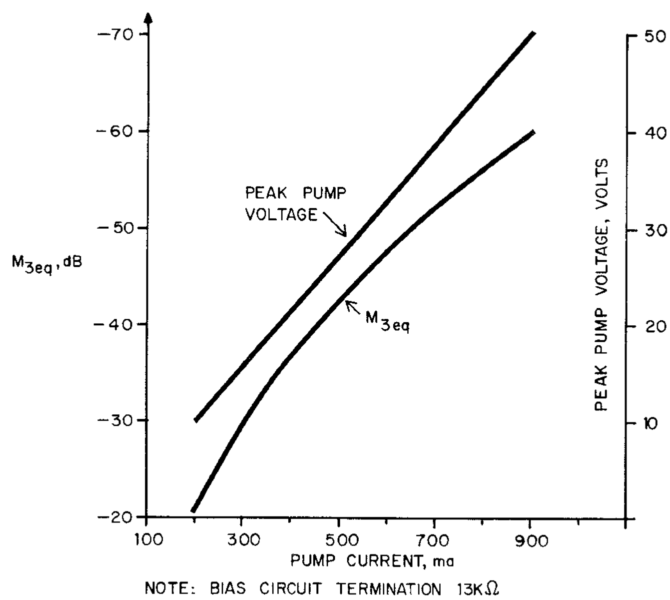


Fig. 3. $M_3$eq variation with pump current.

plus and minus the pump frequency, were all $10^5$ kΩ. Only the real part was varied, although the program will handle complex impedances. The figure shows that the bias circuit is the most sensitive termination, followed by the second-order termination above the input port; the lower sideband had almost no effect on $M_{3eq}(>-140)$ and is not shown. The pump circuit impedance has minimal effect. These results are interesting because to maintain minimal gain shape across the output, it is customary to use a broadband termination. These results show that this results in a minimum penalty on intermodulation. Fig. 3 shows the effect of increased pumping with the bias circuitry at 13 kΩ. There is about a 12-dB improvement for each additional two vars of reactive pump power.

## VIII. Conclusions

The time-varying Volterra series has been developed for an upconverter as a two-terminal periodic nonlinearity and the formulation applied to an abrupt junction varactor upconverter. The results agree in a limiting case with previous work of Gardiner and Ghobrial. Specifically, their equation 46 may be derived from the equations in Section V by assuming all ports are open-circuited except for the signal, pump, and output and letting $b_{11}$, the conversion elastance, tend toward zero. This last condition is equivalent to their assumption of very large pump current and low gain. This highlights the generality of this method, since all ports may have noninfinite terminations.

The example of the upconverter in Section VII shows the importance of the terminations in achieving highly linear operation. Figs. 2 and 3 show that to avoid large pump powers and yet achieve linearities of $-30$ dB or better, the bias port termination and the termination for the sum frequencies of the input should be in the tens of kilohm range.

For the special case of the abrupt junction diode ($n=1/2$), the results of Appendix I and Section V-C show that only second-order intermodulation produces intermodulation noise at the output. Furthermore, the results of Section V-B show that these second-order sources are controlled by the impedances at the second-order intermodulation frequencies (bias, pump, etc.), which are not usually at signal frequencies. Thus, in theory, it is possible to reduce the intermodulation noise for this converter to zero; however, as pointed out above, the impedance levels needed to achieve this situation at microwave frequencies are impractical.

Appendix I provides results to calculate time-varying diode parameters for diffuse junction varactors ($n \neq 1/2$). Of course, additional terms would have to be included in the equations of Section V. Gain compression can be calculated from these results by finding the value of the third-order kernel for a single-frequency input. This term plus the linear term then give the total output and the gain can be calculated directly. Likewise, intermodulation current in the pump port due to the signal may be calculated from the second-order kernel.

## Appendix I
## Coefficients for Pumped Diode

In this Appendix, a model for a varactor diode being pumped from a local oscillator source will be developed as a time-varying nonlinear capacitance. The development follows that of [11] with minor changes and holds for a diode pumped nearly into conduction. The fully driven case is important since the most linear operation occurs in that region. Results are given for coefficients up to the third harmonic of the pump, well beyond those needed for the abrupt junction case, so that a more general diode characteristic can be handled with appropriate changes in (5-1)–(5-5).

### *A. Diode v-q Characteristic*

The pump current through the diode is known and is

$$i_p(t) = -I_p \sin \omega_p t \tag{I-1}$$

where $\omega_p$ is the pump radian frequency,

$I_p$ is the peak pump current.

It is assumed that the bias circuit resistance is much higher than the diode reactance at the pump frequency, so that all the pump current goes through the diode.

The diode capacitance is

$$C(v) = \frac{C(0)}{\left(1 - \frac{v}{\phi}\right)^n} \tag{I-2}$$

where $C(0)$ is the diode capacitance at zero volts, $\phi$ is the contact potential, (usually less than 1), and $n$ is a dimensionless diode parameter.

Integration yields, for the total diode voltage as a function of the total diode charge,

$$v(t) = \phi\left\{1 - \left[1 - \frac{(1-n)q_T}{\phi C(0)}\right]^{1/(1-n)}\right\}$$
$$= f(q_T) \tag{I-3}$$

where $v(t)$ is positive for conduction and $q_T$ is the total junction charge.

### *B. Diode Model*

The total junction charge is composed of three parts: a constant charge due to the bias voltage and any additional charge from pumping; the variable charge due to the pumping current; and the charge due to the signal currents. Call these charges $Q$, $q_p$, and $q$, respectively:

$$q_T = Q + q_p + q. \tag{I-4}$$

From I-1,

$$q_p(t) = A \cos \omega_p t \tag{I-5}$$
$$A = \frac{I_p}{\omega_p}.$$

Now expand (1-3) about the signal charge:

$$v(t) = f(q_T : q=0)$$
$$+ \left.\frac{\partial f(q_T)}{\partial q_T}\right|_{q=0} q + \frac{1}{2!}\left.\frac{\partial^2}{\partial q_T^2} f(q_T)\right|_{q=0} q^2 + \cdots \tag{I-6}$$
$$= b_0(t) + b_1(t)q + b_2(t)q^2 + \cdots. \tag{I-7}$$

The $b$'s are functions of $Q$ and $q_p$ and so are functions of time because $q_p$ is; moreover, since $q_p(t)$ is periodic, so are the $b$'s and they may be expanded in Fourier series:

$$b_0(t) = \sum_{n=0}^{\infty} b_{0n} \cos n\omega_p t. \tag{I-8a}$$

Generally,

$$b_i(t) = \sum_{j=0}^{\infty} b_{ij} \cos j\omega_p t. \tag{I-8b}$$

Knowledge of the coefficients in this series ($b_{ij}$) is then a complete model for the diode. In the next two sections these coefficients will be found in terms of the static parameters of the diode.

### *C. Coefficients in Terms of Peak Pump Amplitude and Fixed Charges*

Rather than attempting to use the integral definition of the Fourier series representation on the partial derivatives of (I-3), we expand the coefficients of (1-7) in power series (the $a_i$ here are not the varistor coefficients of (2-2) of the text):

$$b_0(t) = V_B + a_1 q_p - a_2 q_p^2 - a_3 q_p^3 - \cdots \tag{I-9a}$$
$$b_1(t) = a_1 - 2a_2 q_p - 3a_3 q_p^2 - \cdots \tag{I-9b}$$
$$b_2(t) = \frac{1}{2!}\left(-2!a_2 - \frac{3!}{1!}a_3 q_p - \frac{4!}{2!}a_4 q_p^2 - \cdots\right) \tag{I-9c}$$
$$b_3(t) = \frac{1}{3!}\left(-3!a_3 - \frac{4!}{1!}a_4 q_p - \frac{5!}{2!}a_5 q_p^2 - \cdots\right) \tag{I-9d}$$

where $V_B$ is the diode voltage corresponding to diode charge $q_B$, the unpumped ($q_p = 0$) bias point, and where we used the fact that

$$b_1[q_p] = \frac{\partial b_0[q_p]}{\partial q_p} \tag{I-10a}$$

$$b_2[a_p];\frac{\partial b_1[q_p]}{\partial q_p} \tag{I-10b}$$

and so on. The signs in the expansion for $b_0$ were chosen for convenience.

The $a_i$ will be found in the next section. Powers of $q_p$ may be expanded into harmonics of $\omega_p$ by using DeMoivre's theorem. Now coefficients for a given harmonic of $\omega_p$ for each $b_i$ may be collected. The final results for the $b_{ij}$ in terms of $A$ and $a_i$ are given in (I-11)–(I-14) for $b_{0j}$, $b_{1j}$, $b_{2j}$, and $b_{3j}$, respectively.

$$b_{00}=V_B-a_2\frac{A^2}{2}-a_4\frac{6}{2\cdot 8}A^4-a_6\frac{20}{2\cdot 32}A^6-a_8\frac{70}{2\cdot 128}A^8-a_{10}\frac{252}{2\cdot 512}A^{10} \tag{I-11a}$$

$$b_{01}=a_1A-a_3\frac{3}{4}A^3-a_5\frac{10}{16}A^5-a_7\frac{35}{64}A^7-a_9\frac{126}{256}A^9 \tag{I-11b}$$

$$b_{10}=a_1-a_3\frac{3}{2}A^2-a_5\frac{5\cdot 6}{2\cdot 8}A^4-a_7\frac{7\cdot 20}{2\cdot 32}A^6-a_9\frac{9\cdot 70}{2\cdot 128}A^8 \tag{I-12a}$$

$$b_{11}=-a_2 2A-a_4\frac{4\cdot 3}{4}A^3-a_6\frac{6\cdot 10}{16}A^5-a_8\frac{8\cdot 35}{64}A^7-a_{10}\frac{10\cdot 126}{256}A^9 \tag{I-12b}$$

$$b_{12}=-a_3\frac{3}{2}A^2-a_5\frac{5\cdot 4}{8}A^4-a_7\frac{7\cdot 15}{32}A^6-a_9\frac{9\cdot 56}{128}A^8 \tag{I-12c}$$

$$2!b_{20}=2a_2-a_4\frac{3\cdot 4}{2}A^2-a_6\frac{6\cdot 5\cdot 6}{2\cdot 8}A^4-a_8\frac{8\cdot 7\cdot 20}{2\cdot 32}A^6-a_{10}\frac{10\cdot 9\cdot 70}{2\cdot 128}A^8 \tag{I-13a}$$

$$2!b_{21}=-a_3 3\cdot 2A-a_5\frac{5\cdot 4\cdot 3}{4}A^3-a_7\frac{7\cdot 6\cdot 10}{16}A^5-a_9\frac{9\cdot 8\cdot 35}{64}A^7 \tag{I-13b}$$

$$2!b_{22}=-a_4\frac{4\cdot 3}{2}A^2-a_6\frac{6\cdot 5\cdot 4}{8}A^4-a_8\frac{8\cdot 7\cdot 15}{32}A^6-a_{10}\frac{10\cdot 9\cdot 56}{128}A^8 \tag{I-13c}$$

$$2!b_{23}=-a_5\frac{5\cdot 4}{4}A^3-a_7\frac{7\cdot 6\cdot 5}{16}A^5-a_9\frac{9\cdot 8\cdot 21}{64}A^7 \tag{I-13d}$$

$$3!b_{30}=-a_3 3\cdot 2-a_5\frac{5\cdot 4\cdot 3}{2}A^2-a_7\frac{7\cdot 6\cdot 5\cdot 6}{2\cdot 8}A^4-a_9\frac{9\cdot 8\cdot 7\cdot 20}{2\cdot 32}A^6 \tag{I-14a}$$

$$3!b_{31}=-a_4 4\cdot 3\cdot 2A-a_6\frac{6\cdot 5\cdot 4\cdot 3}{4}A^3-a_8\frac{8\cdot 7\cdot 6\cdot 10}{16}A^5-a_{10}\frac{10\cdot 9\cdot 8\cdot 35}{64}A^7 \tag{I-14b}$$

$$3!b_{32}=-a_5\frac{5\cdot 4\cdot 3}{2}A^2-a_7\frac{7\cdot 6\cdot 5\cdot 4}{8}A^4-a_9\frac{9\cdot 8\cdot 7\cdot 15}{32}A^6 \tag{I-14c}$$

$$3!b_{33}=-a_6\frac{6\cdot 5\cdot 4}{4}A^3-a_8\frac{8\cdot 7\cdot 6\cdot 5}{16}A^5-a_{10}\frac{10\cdot 9\cdot 8\cdot 21}{64}A^7 \tag{I-14d}$$

$$3!b_{34}=-a_7\frac{7\cdot 6\cdot 5}{8}A^4-a_9\frac{9\cdot 8\cdot 7\cdot 6}{32}A^6. \tag{I-14e}$$

*D. The $a_i$ in Terms of Static Parameters*

Since $q_T$ is the sum of three charges, we have

$$\left.\frac{\partial f(q_T)}{\partial q_T}\right|_{q=0,q_p=0}=\left.\frac{\partial f(q_T,q=0)}{\partial q_T}\right|_{q_p=0}=\left.\frac{\partial b_0(t)}{\partial q_p}\right|_{q_p=0}=a_1. \tag{I-15}$$

Thus differentiating (I-3) and setting $q$ and $q_p$ to zero gives

$$\frac{1}{C(0)}\left[1-\frac{(1-n)Q}{\phi C(0)}\right]^{n/(1-n)}=a_1. \tag{I-16}$$

Let $C_B$ be the capacitance of the diode at voltage $V_B$ with no pumping or signal and let charge $q_B$ be the charge at $V_B$ with no pumping or signal. Then using (I-2) in (I-16),

$$a_1=\frac{1}{C_BE} \tag{I-17}$$

where

$$E=\left[\frac{1-\dfrac{(1-n)q_B}{\phi C(0)}}{1-\dfrac{(1-n)Q}{\phi C(0)}}\right]^{n/(1-n)} \tag{I-18}$$

Using the factor $E$, the pumped diode parameters may be expressed in terms of the more conveniently measured unpumped parameters. Note that charges are negative for negative voltage, so $E<1$. In particular, if $q_0$ is the charge at zero bias,

$$\frac{C_j(0)}{C_B}=\left[1-\frac{V_B}{\phi}\right]^n \tag{I-19}$$

$$=\left[1-\frac{(1-n)q_0}{\phi C_j(0)}\right]^{n/(1-n)} \tag{I-20}$$

from which (I-17) may be found directly. Successive differentiations of (I-3) then give

$$a_1=\frac{1}{C_B}\frac{1}{E} \tag{I-21a}$$

$$a_2=\frac{n}{2!(\phi-V_B)C_B^2}E^{(1/n)-2} \quad \text{(I-21b)}$$

$$a_3=\frac{n(1-2n)}{3!(\phi-V_B)^2C_B^3}E^{(2/n)-3} \quad \text{(I-21c)}$$

$$a_4=\frac{n(1-2n)(2-3n)}{4!(\phi-V_B)^3C_B^4}E^{(3/n)-4} \quad \text{(I-21d)}$$

$$a_5=\frac{n(1-2n)(2-3n)(3-4n)}{5!(\phi-V_B)^4C_B^5}E^{(4/n)-5}. \quad \text{(I-21e)}$$

To determine $E$, let $V_0$ be the voltage corresponding to charge $Q$ on the diode. Then

$$V_0=\frac{1}{T}\int_{-T/2}^{T/2}v(t)\,dt=b_{00} \quad \text{(I-22)}$$

for no signal charge. However, using (I-3), (I-2), and (I-18) we have

$$V_0=V_B+\left(1-\frac{1}{E^{1/n}}\right)(\phi-V_B). \quad \text{(I-23)}$$

Using (I-11a),

$$0=\left(1-\frac{1}{E^{1/n}}\right)(\phi-V_B)+a_2\frac{A^2}{2}+a_4\frac{3}{8}A^4+a_6\frac{5}{16}A^6+a_8\frac{35}{128}A^8+a_{10}\frac{63}{256}A^{10}. \quad \text{(I-24)}$$

Since $E$ is usually close to 1, this equation may be solved iteratively for $E$ using 1 as a starting value.

For the abrupt junction varactor, $n=1/2$ and

$$a_k=0, \qquad k\geqslant 3.$$

The equation for $E$ can be solved directly and is

$$E=\frac{1}{\sqrt{1+\dfrac{A^2}{8(\phi-V_B)^2C_B^2}}}. \quad \text{(I-25)}$$

The Fourier coefficients are then

$$b_{00}=V_0 \qquad b_{01}=a_1A$$
$$b_{10}=a_1 \qquad b_{11}=-2a_2A.$$
$$b_{20}=-a_2$$

The coefficient $b_{20}$ is the sole source of nonlinear behavior, but it cannot be eliminated without eliminating the frequency conversion properties of the diode ($b_{11}=0$). So we have, for the abrupt junction varactor,

$$v(t)=V_0+a_1A\cos\omega_p t+a_1q-2a_2Aq\cos\omega_p t-a_2q^2. \quad \text{(I-26)}$$

The term in $\cos 2\omega_p t$ is assumed filtered out. The advantage of this formulation is that the nonlinear response of the diode as a frequency changer may be calculated at any pumping level, including near maximum loads where best linearity is achieved. For the special case of the abrupt junction diode, the series terminates at the second harmonic.

### E. Units

A convenient, consistent set of units for upconverters used between IF frequencies in the megahertz range to microwaves is

| | |
|---|---|
| potential: | volts |
| charge: | picocoulombs |
| current: | milliamperes |
| capacitance: | picofarads |
| frequency: | gigahertz (gigaradians/second for radian frequency) |
| time: | nanoseconds. |

These are used for quantities in the DATA statements for the computer program. The calculations in the program are independent of units selection.

From the expression for $v=f(q)$:

| Coefficient | Dimensions |
|---|---|
| $b_{0i}$ | volts |
| $b_{1i}$ | elastance (inverse capacitance) |
| $b_{2i}$ | volts/(charge)$^2$ or elastance/charge |
| $b_{3i}$ | volts/(charge)$^3$ or elastance/(charge)$^2$ |

## References


[1] D. R. Anderson and B. J. Leon, "Nonlinear distortion and truncation errors in frequency converters and parametric amplifiers," *IEEE Trans. Circuit Theory*, vol. CT-12, pp. 314–321, Sept. 1965.

[2] S. M. Perlow and B. S. Perlman, "A large signal analysis leading to intermodulation distortion prediction in abrupt junction varactor upconverters," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 820–827, Nov. 1965. (See also comments by Grayzel in Mar. 1967.)

[3] E. G. Schwarz and C. E. Nelson, "High-frequency receiver investigation—Varactor upconverter nonlinear analysis," General Electric Technical Information Series, R70ELS-30, Feb. 1970.

[4] J. G. Gardiner and S. I. Ghobrial, "Distortion performance of the abrupt-junction current-pumped varactor frequency converter," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 741–749, Sept. 1971. (See also the correction in Oct. 1971.)

[5] S. O. Rice, "Volterra systems with more than one input port—Distortion in a frequency converter," *The Bell System Technical Journal*, vol. 52, no. 8, pp. 1255–1270, Oct. 1973.

[6] R. H. Flake, "Volterra series representation of time-varying nonlinear systems," in *Proc. Second Int. Congress of IFAC on Automatic Control* (Basel, Switzerland) 1963, paper no. 408/1.

[7] P. Penfield, Jr. and R. P. Rafuse, *Varactor Applications. M.I.T.*, 1962.

[8] L. A. Zadeh, "Frequency analysis of variable networks," *Proc. I.R.E.*, pp. 291–299, Mar. 1950.

[9] A. Korpel and V. Ramaswamy, "Input conductance of a four-frequency parametric up-converter," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 96-106, Jan. 1965.

[10] Members of the Staff of Bell Telephone Laboratories, *Transmission Systems for Communications*, 4th Edition, Western Electric Co.

[11] C. E. Nelson, "The theoretical nonlinear distortion in varactor upper-sideband upconverters," Internal Document of General Electric Electronics Laboratory, TIS No. R66ELS-114, Nov. 1966.